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Cosmological Implication of the Trace Anomaly

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We establish a connection between the trace anomaly and thermal radiation in the standard cosmology. This is done by solving the covariant conservation equation of the stress tensor associated with a conformally invariant quantum scalar field. The solution corresponds to thermal radiation with a temperature which is given in terms of a cut-off time excluding the spacetime regions very close to the initial singularity. We discuss the interrelation between this result and the result obtained in a two-dimensional Schwarzschild spacetime.

KEY WORDS: cosmology; trace anomaly.

1. INTRODUCTION

In quantum field theory of curved spacetime, matter is described by quantum field theory while the gravitational field itself is regarded as a classical object. In this framework, the stress tensor associated with a quantum field is not well defined and contains singularities. Renormalization prescriptions (Birrell and Davies, 1982) are usually used to obtain a meaningful expression for the stress tensor of a quantum field. One of the most remarkable consequences of these prescriptions is the so-called trace anomaly (Wald, 1977, 1978). This means that the trace of the quantum stress tensor of a conformal invariant field obtains a nonzero expression while the trace of the classical stress tensor vanishes identically.

In a two-dimensional Schwarzschild spacetime there is a close correspondence between the trace anomaly and Hawking radiation (Hawking, 1975), namely the thermal radiation associated with a black hole at null infinity (Christensen and Fulling, 1977). The radiation has a temperature $T = (4\pi k_B R_S)^{-1}$ where k_B is Boltzman's constant and R_S is the Schwarzschild radius of a black hole. Here the length scale R_S may be interpreted as a cut-off length excluding the intrinsic singularity in the interior region of the Schwarzschild solution. In this sense, the

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temperature of the Hawking radiation is given in terms of a cut-off length, R_S, which is characteristic of the Schwarzschild spacetime.

Following the result obtained in the Schwarzschild spacetime we intend here to relate trace anomaly to properties of a background heat bath in the standard cosmology. We do this in two parts. In Section 2, we first use a two-dimensional cosmological model to find the most general solution of the covariant conservation equation of the quantum stress tensor associated with a conformally invariant scalar field. We show that the solution corresponds to an equilibrium gas with a temperature $T \propto (k_B t_c^{-1})$ with t_c being a cut-off time. This cut-off time is defined to avoid the spacetime regions in which the semiclassical investigations are not valid. In Section 3, we show that contrary to the Schwarzschild spacetime, the symmetries of the standard cosmology allow us to solve the covariant conservation equation in four dimensions. We solve the conservation equations for spatially flat Friedmann–Robertson–Walker (FRW) spacetime. At late times, the solution corresponds to an equilibrium heat bath with the same temperature and physical properties of the temperature obtained in the two-dimensional case. In Section 4, we outline our results.

Throughout the following we use units in which $\hbar = c = 1$ and the signature is $(- + ++)$.

2. THE MODEL

Let us begin with the results of renormalization of stress tensor T_μ^ν of a quantum scalar field coupled with a two-dimensional gravitational background, namely,

$$\nabla_\nu T_\mu^\nu = 0 \tag{1}$$

$$T_\mu^\mu = \frac{1}{24\pi} R \tag{2}$$

where ∇_ν denotes a covariant differentiation and R is the curvature scalar. The first equation is a covariant conservation law and the second one indicates an anomalous trace emerging from the renormalization process. We first solve the conservation equation (1) for a two-dimensional cosmological model described by the metric

$$ds^2 = -dt^2 + a^2(t)\,\mathrm{d}x^2 \tag{3}$$

where $a(t)$ is the scale factor. This is a two-dimensional analog of the spatially flat Friedmann–Robertson–Walker (FRW) spacetime. The metric (3) can be written in a conformally flat form

$$ds^2 = a^2(\tau)(-d\tau^2 + dx^2) \tag{4}$$

where

$$\tau = \int \frac{dt}{a(t)} \tag{5}$$

is the conformal time. For the metric (4), the nonvanishing Christoffel symbols are given by

$$\Gamma^{\tau}_{\tau\tau} = \Gamma^{\tau}_{xx} = \Gamma^{x}_{\tau x} = \frac{1}{a}\frac{da}{d\tau} \tag{6}$$

In the spacetime described by (4), all components of the stress tensor can be only functions of time. Using this fact, the equation (1) takes the form

$$\frac{d}{d\tau}T^{\tau}_{\tau} + \Gamma^{x}_{x\tau}T^{\tau}_{\tau} - \Gamma^{x}_{x\tau}T^{x}_{x} = 0 \tag{7}$$

$$\frac{d}{d\tau}T^{\tau}_{x} + \Gamma^{\tau}_{\tau\tau}T^{\tau}_{x} - \Gamma^{\tau}_{xx}T^{x}_{\tau} = 0 \tag{8}$$

For the off-diagonal elements of the stress tensor we have $T^{\tau}_{x} = -T^{x}_{\tau}$. On the other hand, one can write $T^{x}_{x} = T^{\mu}_{\mu} - T^{\tau}_{\tau}$. These relations among different components of the stress tensor together with (5) and (6) allow us to write (7) and (8) in the form

$$\frac{d}{dt}\left(a^2 T^{\tau}_{\tau}\right) = a\frac{da}{dt}T^{\mu}_{\mu} \tag{9}$$

$$\frac{d}{dt}\left(a^2 T^{\tau}_{x}\right) = 0 \tag{10}$$

Equation (10) immediately gives

$$T^{\tau}_{x} = \alpha\, a^{-2} \tag{11}$$

with α being an integration constant. The solution of the equation (9) is

$$T^{\tau}_{\tau} = a^{-2}(h + \beta) \tag{12}$$

where

$$h = \int_{t_c}^{t} T^{\mu}_{\mu}(t')\, \frac{da(t')}{dt'}\, a(t')\, dt' \tag{13}$$

$$\beta = a^2(t_c)\, T^{\tau}_{\tau}(t_c) \tag{14}$$

and t_c is an arbitrary time scale. Given a time scale t_c, the function h incorporates the corresponding contribution of the trace T^{μ}_{μ} in the stress tensor T^{ν}_{μ}. The introduction of the cut-off time t_c is a mandate in order to exclude in the definition of h the contribution of the trace very close to the early time singularity. In fact in that region an accurate description of quantum gravity is needed and the semiclassical approach is no longer valid.

In a homogeneous and isotropic universe, we require that $\alpha = 0$. This implies that the stress tensor has vanishing off-diagonal elements. In this case we obtain

$$T_\mu^\nu = a^{-2}(h+\beta)\begin{pmatrix} 1 & 0 \\ 0 & q-1 \end{pmatrix} \tag{15}$$

where

$$q = \frac{T_\mu^\mu}{a^{-2}(h+\beta)} \tag{16}$$

We are particularly interested in the late-time configuration of the stress tensor. It obviously depends on the explicit form of the scale factor. Thus for studying T_μ^ν at late times, we first assume that the scale factor follows a power law expansion

$$a = a_0\left(\frac{t}{t_0}\right)^n \tag{17}$$

with t_0 being the present age of the universe. We then use the explicit form of the trace anomaly for the metric (4)[2] to write (13) and (16) in the form

$$h(t \to t_0) = \frac{n^2\, a_0^2\, t_0^{-2}}{24\pi}\left\{1 - l^{2(n-1)}\right\} \tag{18}$$

$$q(t \to t_0) = \frac{2(n-1)}{n}\left\{1 + \frac{n-2}{n}\, l^{2(n-1)}\right\}^{-1} \tag{19}$$

where $l = \frac{t_c}{t_0}$. Putting the relations (14) and (18) into (15), we obtain

$$T_\mu^\nu(t \to t_0) = \frac{n^2\, t_0^{-2}}{24\pi}\left\{1 + \frac{n-2}{n} l^{2(n-1)}\right\}\begin{pmatrix} 1 & 0 \\ 0 & q(t \to t_0) - 1 \end{pmatrix} \tag{20}$$

One should note that the cut-off time t_c is much smaller than t_0 so that $l \ll 1$. One therefore infers that $l^{2(n-1)} \gg 1$ for $n < 1$. In this case, the relation (19) indicates that $q(t \to t_0) \ll 1$. With this approximation the relation (20) takes the form

$$T_\mu^\nu(t \to t_0) = \frac{n(2-n)\, l^{2n}\, t_c^{-2}}{24\pi}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{21}$$

When one compares (21) with the stress tensor of an equilibrium gas, namely

$$\frac{\pi}{6}(k_B T)^2\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{22}$$

[2] See Appendix

one concludes that at late times the stress tensor T_μ^ν describes an equilibrium gas with temperature

$$T = \frac{1}{2\pi}\sqrt{n(2-n)}\, l^n\, (k_B t_c)^{-1} \tag{23}$$

It is interesting to compare (23) with the result obtained in the Schwarzschild spacetime. In that case, the temperature of the thermal radiation is given in terms of R_S^{-1}. The Schwarzschild radius R_S may be interpreted as a cut-off length that disjoints the interior and the exterior regions of the Schwarzschild solution. In principle, this is very similar to the case of the standard cosmology. In this case, the temperature of the equilibrium gas is given in terms of t_c^{-1}. Here the cut-off time t_c excludes the early stages of evolution of the universe in which the semiclassical calculations cannot be applied.

3. THE FOUR-DIMENSIONAL CASE

In the standard cosmology, the universe is assumed to be isotropic in all points of spacetime. This is a larger symmetry with respect to the Schwarzschild spacetime that allows us to generalize our results obtained in the previous section to four dimensions. In a four-dimensional spacetime, the analog of the equation (2) is

$$T_\mu^\mu = -2v_1(x) \tag{24}$$

where

$$v_1(x) = \frac{1}{720}(\Box R - R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta}) \tag{25}$$

Here $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$, $R_{\mu\nu}$ and R are the first and the second contraction of the Riemann curvature tensor $R_{\mu\nu\gamma\delta}$, respectively. We intend to solve the conservation equation (1) for the spatially flat FRW metric described by

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \tag{26}$$

It can be written in a conformally flat form

$$ds^2 = a^2(\tau)(-d\tau^2 + dx^2 + dy^2 + dz^2) \tag{27}$$

where the conformal time τ is given by (5). The nonzero Christoffel symbols of the metric (27) are

$$\Gamma^\tau_{\tau\tau} = \Gamma^\tau_{xx} = \Gamma^\tau_{yy} = \Gamma^\tau_{zz} = \Gamma^x_{\tau x} = \Gamma^y_{\tau y} = \Gamma^z_{\tau z} = \frac{1}{a}\frac{da}{d\tau} \tag{28}$$

The homogeneity and the isotropy of the universe imply that all components of the stress tensor T_μ^ν are space independent and can be only functions of time. We use this fact to write the time component of the conservation equation (1)

$$\frac{d}{d\tau}T^{\tau}_{\tau}+\Gamma^{x}_{x\tau}T^{\tau}_{\tau}+\Gamma^{y}_{y\tau}T^{\tau}_{\tau}+\Gamma^{z}_{z\tau}T^{\tau}_{\tau}-\Gamma^{x}_{x\tau}T^{x}_{x}-\Gamma^{y}_{y\tau}T^{y}_{y}-\Gamma^{z}_{z\tau}T^{z}_{z}=0 \tag{29}$$

There are also three space components

$$\frac{d}{d\tau}T^{\tau}_{x}+\Gamma^{\tau}_{\tau\tau}T^{\tau}_{x}+\Gamma^{x}_{x\tau}T^{\tau}_{x}+\Gamma^{y}_{y\tau}T^{\tau}_{x}+\Gamma^{z}_{z\tau}T^{\tau}_{x}-\Gamma^{x}_{\tau x}T^{\tau}_{x}-\Gamma^{\tau}_{xx}T^{x}_{\tau}=0 \tag{30}$$

$$\frac{d}{d\tau}T^{\tau}_{y}+\Gamma^{\tau}_{\tau\tau}T^{\tau}_{y}+\Gamma^{x}_{x\tau}T^{\tau}_{y}+\Gamma^{y}_{y\tau}T^{\tau}_{y}+\Gamma^{z}_{z\tau}T^{\tau}_{y}-\Gamma^{x}_{\tau x}T^{\tau}_{x}-\Gamma^{\tau}_{xx}T^{x}_{\tau}=0 \tag{31}$$

$$\frac{d}{d\tau}T^{\tau}_{z}+\Gamma^{\tau}_{\tau\tau}T^{\tau}_{z}+\Gamma^{x}_{x\tau}T^{\tau}_{z}+\Gamma^{y}_{y\tau}T^{\tau}_{z}+\Gamma^{z}_{z\tau}T^{\tau}_{z}-\Gamma^{x}_{\tau x}T^{\tau}_{x}-\Gamma^{\tau}_{xx}T^{x}_{\tau}=0 \tag{32}$$

For other components of the stress tensor, we have

$$T^{x}_{y}=T^{y}_{x}=T^{x}_{z}=T^{z}_{x}=T^{y}_{z}=T^{z}_{y}=0 \tag{33}$$

$$T^{x}_{x}=T^{y}_{y}=T^{z}_{z} \tag{34}$$

$$T^{\tau}_{x}=T^{\tau}_{y}=T^{\tau}_{z}=-T^{x}_{\tau}=-T^{y}_{\tau}=-T^{z}_{\tau} \tag{35}$$

$$T^{x}_{x}=\frac{1}{3}\left(T^{\mu}_{\mu}-T^{\tau}_{\tau}\right) \tag{36}$$

If we use the relations (5) and (28) together with (33)–(36), the equations (29) and (30) take the form[3]

$$\frac{d}{dt}\left(a^{4}T^{\tau}_{\tau}\right)=a^{3}\frac{da}{dt}T^{\mu}_{\mu} \tag{37}$$

$$\frac{d}{dt}\left(a^{4}T^{\tau}_{x}\right)=0 \tag{38}$$

These equations yield

$$T^{\tau}_{x}=\delta\, a^{-4} \tag{39}$$

$$T^{\tau}_{\tau}=a^{-4}(H+\gamma) \tag{40}$$

where

$$H=\int_{t_c}^{t}T^{\mu}_{\mu}(t')\,\frac{da(t')}{dt'}\,a^{3}(t')\,dt' \tag{41}$$

$$\gamma=a^{4}(t_{\mathrm{c}})\,T^{\tau}_{\tau}(t_{\mathrm{c}}) \tag{42}$$

[3] Due to (35) the same equations like (38) hold for the other two components T^{τ}_{y} and T^{τ}_{z}.

and δ is an integration constant. If we set $\delta = 0$ and use the trace condition (36), we obtain

$$T_{\mu}^{\nu} = a^{-4}(H+\gamma)\,\mathrm{diag}\left\{1,\ \frac{1}{3}(Q-1)\,,\ \frac{1}{3}(Q-1)\,,\ \frac{1}{3}(Q-1)\right\} \tag{43}$$

where

$$Q = \frac{T_{\mu}^{\mu}}{a^{-4}(H+\gamma)} \tag{44}$$

We put the explicit form of the trace anomaly (24) for the metric (27)[4] into (41) and (44) to obtain H and Q in asymptotic times

$$H(t \to t_0) = \frac{1}{120}\, a_0^4\, n^2\, (n^2 - 6n + 3)\, t_0^{-4}\left\{1 - l^{4(n-1)}\right\} \tag{45}$$

$$Q(t \to t_0) = \frac{4(n-1)}{n}\left\{1 + \frac{(3n-4)}{n}\, l^{4(n-1)}\right\}^{-1} \tag{46}$$

The same approximation used in the two-dimensional case, namely that $l^{4(n-1)} \gg 1$ for $n < 1$ results in $Q(t \to t_0) \ll 1$. Within this approximation and using (42) and (45), (43) reduces to

$$T_{\mu}^{\nu}(t \to t_0) = \lambda\, t_c^{-4}\, \mathrm{diag}\left\{-1,\ \frac{1}{3}\,,\ \frac{1}{3}\,,\ \frac{1}{3}\right\} \tag{47}$$

where λ is a dimensionless constant. This corresponds to the stress tensor of a radiation with energy density $\rho \propto t_c^{-4}$. There is a specific relation between the energy density and temperature $\rho \propto (k_B T)^4$ if the radiation is thermally distributed. Therefore the temperature of the radiation is given by $T \propto (k_B\, t_c)^{-1}$, namely, like the two-dimensional case, the temperature is proportional to the inverse of the cut-off time.

4. CONCLUDING REMARKS

We have investigated the stress tensor of a conformally invariant quantum scalar field in a homogeneous and isotropic cosmology. The covariant conservation equation of the stress tensor is solved in two different cases, namely in two and four dimensional spacetimes. In both cases, we have shown that the late time configuration of the stress tensor can be connected to the stress tensor of an equilibrium gas with a temperature that is given in terms of a cut-off time t_c. This cut-off is introduced to exclude the effects of early-time cosmology, in which a full theory of quantum gravity holds, from our semiclassical calculations.

[4] See Appendix.

We would like to emphasize the following remarks:

(a) In a cosmological context, we have developed a connection between trace anomaly and a thermal radiation in two and four dimensions.
(b) It is well known that the thermodynamic properties of the Schwarzschild spacetime can be extended to the cosmological models with a repulsive cosmological term (Gibbons and Hawking, 1977). In such cosmological models each observer will detect an isotropic background of thermal radiation. This result shows resemblance to the result obtained in the present work. In the absence of a cosmological constant in our model universe, this resemblance implies that the late time configuration of the anomalous trace in a cosmological context has a nonzero contribution to the cosmological constant. This feature of the trace anomaly has been already studied in a different framework in (Salehi and Bisarb, 2000; Salehi et al., 2000).

APPENDIX

We consider two metric tensors $\bar{g}_{\mu\nu}$ and $g_{\mu\nu}$ which are conformally related, namely

$$\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \tag{A.1}$$

where Ω is a smooth dimensionless spacetime function. If $g_{\mu\nu}$ describes the Minkowski spacetime, $\eta_{\mu\nu}$, the metric $\bar{g}_{\mu\nu}$ is said to be conformally flat.

In a two-dimensional spacetime, any metric tensor can take a conformally flat form. In this case the curvature scalar is given by (Wald, 1984)

$$\bar{R} = -2(\Omega^{-2}\Box_\eta \ln \Omega) \tag{A.2}$$

where $\Box_\eta$ is the d'Alambertian operator in Minkowski spacetime. Substituting this into the relation (2) and noting the fact that $\Omega(x) = a(\tau)$, we obtain for the trace anomaly

$$T^\mu_\mu(\bar{g}_{\mu\nu}) = \frac{1}{12\pi}\,\frac{1}{a}\,\frac{d^2 a}{dt^2} \tag{A.3}$$

in which we have used (5) to express the derivative of the scale factor with respect to the coordinate time t. If one uses the explicit form of the scale factor (17) in the relation (A.3), one arrives at

$$T^\mu_\mu(\bar{g}_{\mu\nu}) = \frac{1}{12\pi}\, n\,(n-1)\,t^{-2} \tag{A.4}$$

In the four-dimensional case, the trace anomaly (24) for the metrics $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ are related by (Brown, 1984)

$$\begin{aligned} T^{\mu}_{\mu}(\bar{g}_{\mu\nu}) = -\frac{1}{360}e^{4\omega}\{\Box R - R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} + 2R\Box\omega \\ + 2R_{;\gamma}\omega^{;\gamma} + 6\Box(\Box\omega) + 8(\Box\omega)^2 - 8\omega_{;\mu\nu}\omega^{;\mu\nu} - 8R_{\mu\nu}\omega^{;\mu}\omega^{;\nu} \\ - 8\omega_{;\gamma}\omega^{;\gamma}\Box\omega - 16\omega_{;\mu\nu}\omega^{;\mu}\omega^{;\nu}\} \end{aligned} \tag{A.5}$$

where $\omega = -\ln\Omega$ and semicolon indicates covariant differentiation. If $g_{\mu\nu} = \eta_{\mu\nu}$, (A.5) reduces to

$$\begin{aligned} T^{\mu}_{\mu}(\bar{g}_{\mu\nu}) = -\frac{1}{180}e^{4\omega}\{3\Box_\eta(\Box_\eta\omega) + 4(\Box_\eta\omega)^2 - 4\omega_{;\mu\nu}\omega^{;\mu\nu} \\ - 4\omega_{;\gamma}\omega^{;\gamma}\Box_\eta\omega - 8\omega_{;\mu\nu}\omega^{;\mu}\omega^{;\nu}\} \end{aligned} \tag{A.6}$$

We may use $\omega = -\ln a$ and (5) to write $T^{\mu}_{\mu}(\bar{g}_{\mu\nu})$ in the form

$$T^{\mu}_{\mu}(\bar{g}_{\mu\nu}) = \frac{1}{60}\left\{\frac{1}{a}\frac{d^4a}{dt^4} + \frac{1}{a^2}\frac{d^2a}{dt^2} + 3\frac{1}{a^2}\frac{da}{dt}\frac{d^3a}{dt^3} - 3\frac{1}{a^3}\frac{d^2a}{dt^2}\left(\frac{da}{dt}\right)^2\right\} \tag{A.7}$$

With the scale factor (17), (A.7) is equivalent to

$$T^{\mu}_{\mu} = \frac{1}{30}n(n-1)(n^2-6n+3)\,t^{-4} \tag{A.8}$$

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